Introduction

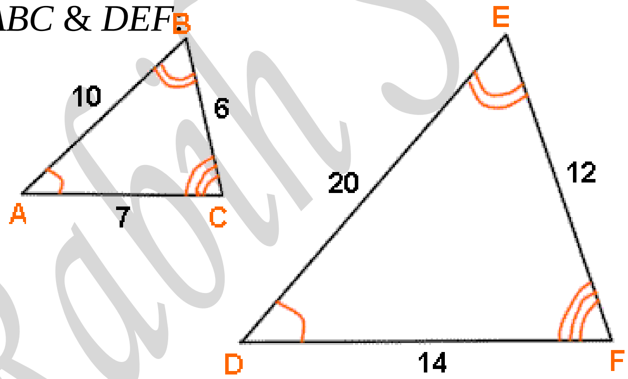
The Cat to the right is an enlargement of the one to the left. They are exactly of the ***same*** shape, but they are *NOT* of the same ***size***.

These cats represent ***similar figures***.

*Objects, such as the above two cats, that have the **same shape**, but do **not** have the **same size**, are called "similar".*

> ***Definition:*** In mathematics, two triangles are said to be ***similar*** if their corresponding (matching) angles are ***congruent*** and the ***ratios*** of their corresponding ***sides*** are ***proportional***.

Consider the two triangles *ABC* & *DEF*



Comparing elements of the above triangles		
Angles	**Ratios of sides**	
$\hat{A} \cong \hat{D}$	$\frac{AB}{DE} = \frac{10}{20} = \frac{1}{2}$	$\Rightarrow \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF} = \frac{1}{2}$
$\hat{B} \cong \hat{E}$	$\frac{AC}{DF} = \frac{7}{14} = \frac{1}{2}$	
$\hat{C} \cong \hat{F}$	$\frac{BC}{EF} = \frac{6}{12} = \frac{1}{2}$	

This ratio is the Scale Factor ***K***

Therefore; Triangles ***ABC*** and ***DEF*** are **similar**.

In symbols: $\Delta ABC \sim \Delta DEF$

Strategies for Proving Triangles Similar

To ***show that*** two ***triangles*** are ***similar***, it is sufficient to use one of the following cases:

CASE-1: *Two angles, postulate*

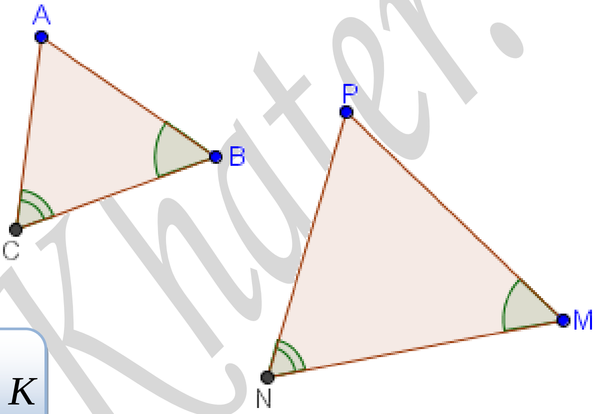
AA	If ***two angles*** of one triangle are congruent (***equal***) to ***two angles*** of the other triangle, then the two triangles are similar.

In the two triangles *ABC* & *PMN* we have:

1- $A\hat{B}C = P\hat{M}N$

2- $B\hat{C}A = M\hat{N}P$

Hence, $\Delta\ ABC \sim \Delta\ PMN$, by "***AA***" postulate.



Ratio of similitude:

$$\frac{ABC}{PMN} \Big| \ \frac{AB}{PM} = \frac{AC}{PN} = \frac{BC}{MN} = K$$

CASE-2: *Side – side angle inside, postulate:*

SAS	If an ***angle*** of one triangle is ***equal*** to an ***angle*** of the other triangle and the ***two*** adjacent ***sides*** of these angles are ***proportional***, then the two triangles are similar.

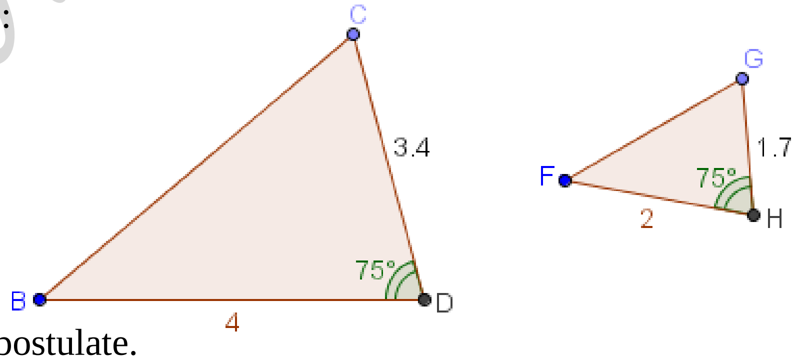
In the two triangles *BCD* & *FGH* we have:

1- $\frac{BD}{FH} = \frac{4}{2} = 2$

2- $B\hat{D}C = F\hat{H}G = 75^{\circ}$

3- $\frac{CD}{GH} = \frac{3.4}{1.7} = 2$

Hence, $\Delta\ BCD \sim \Delta\ FGH$, by "S A S" postulate.



Ratio of similitude:

$$\frac{BCD}{FGH} \Big| \ \frac{BC}{FG} = \frac{BD}{FH} = \frac{CD}{GH} = 2 = K$$

Enlargement Factor
Since $K > 1$.

$$\frac{ABC}{PMN} \Big| \ \frac{AB}{PM} = \frac{AC}{PN} = \frac{BC}{MN} = K$$

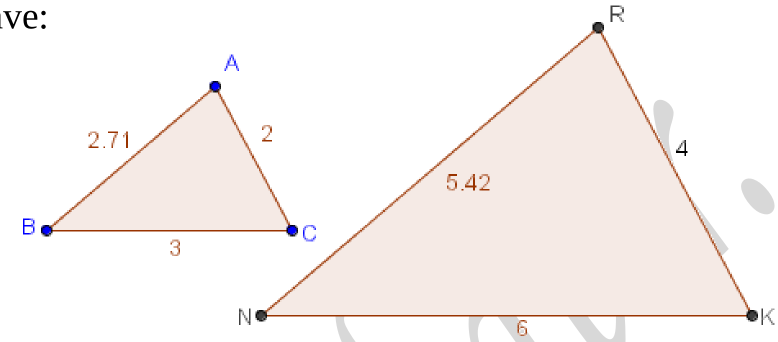
Keep an open mind!

Remember that there might be more than one way to arrive at an answer!

CASE-3: Side – side – Side, postulate:

SSS	If ***three sides*** of one triangle are ***proportional*** to ***three sides*** of the other triangle, then the two triangles are similar.

In the two triangles ABC & RNK we have:



1- $\frac{AB}{RN}=\frac{2.71}{5.42}=\frac{1}{2}$

2- $\frac{AC}{RK}=\frac{2}{4}=\frac{1}{2}$

3- $\frac{BC}{NK}=\frac{3}{6}=\frac{1}{2}$

Hence, $\Delta\ ABC \sim \Delta\ RNK$, by "***SSS***" postulate.

Ratio of similitude:

$$\frac{ABC}{RNK}\Bigg|\ \frac{AB}{RN}=\frac{AC}{RK}=\frac{BC}{NK}=\frac{1}{2}=K$$

Reduction Factor
Since $K<1$

How to Find Similarity Ratio?

$$\frac{ABC}{PMN}\Bigg|\ \frac{AB}{PM}=\frac{AC}{PN}=\frac{BC}{MN}=K$$

To find similarity ratio:

1. ***Match pairs of corresponding sides.***
2. ***Match pairs of corresponding angles.***
3. ***Divide matched pairs of corresponding sides and angles to get your ratio***.

Attention!!!

↳ ***The scale factor K is said to be a ratio of:***

Case	*Condition (if)*
Enlargement	$K>1$
Reduction:	$K<1$
Congruency:	$K=1$

↳ ***Why similar triangles?***

We use similar triangles to:

a) Find ***missing lengths***.
b) Find the ***enlargement*** or ***reduction*** of an object.
c) Find an ***algebraic relation*** between sides of two triangles.
d) Find ***height of tall objects*** such as trees, buildings…

Interesting Ratios to Study in Similar Triangles

1. Ratio of Perimeters, Altitudes, Medians, and Angle Bisectors

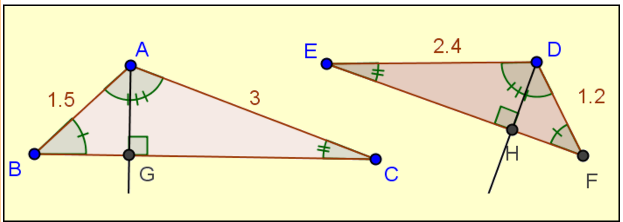
If two triangles are **similar**, then their corresponding sides, altitudes, medians, angle bisectors and perimeters are all divided in the same ratio.

In other words, we can always include ratios of ***altitudes***, ***medians***, ***angle bisectors*** and ***perimeters*** in ratio of similitude.

In a mathematical statement,

$$\text{If } \frac{s_1}{s_1'} = \frac{s_2}{s_2'} = \frac{s_3}{s_3'} = k \text{ then } \frac{h}{h'} = k$$

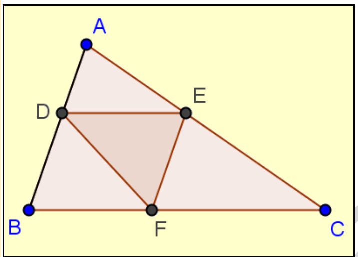
Example-1:



Consider the triangles *ABC* & *DEF*.
Find the ratio of their altitudes.

Answers: $\frac{AG}{DH} = \frac{5}{4}$.

Example-2:



The ratio of sides of the two similar triangles *ABC* & *DEF* is 4 : 9.

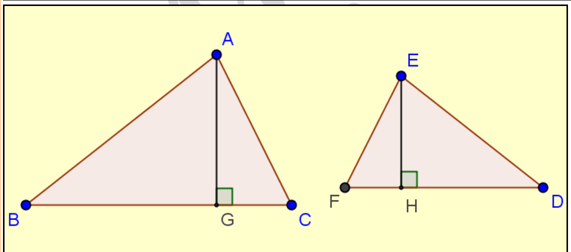
a) What does the ratio 4 : 9 represent? Justify.

b) What is the ratio of their perimeters?

Answers: a) Ratio of reduction. b) 4 : 9

2. Ratio of Areas

Example-3:



Given that triangles ABC & DEF are similar.

Prove that $\frac{Area\,\Delta\,ABC}{Area\,\Delta\,DEF} = \left(\frac{BC}{DF}\right)^2$

Conclusion:

If two triangles are **similar**, then the ratio of their ***areas*** is equal to the ***square*** of the ratio of their corresponding sides.

In a mathematical statement,

$$\text{If } \frac{s_1}{s_1'} = \frac{s_2}{s_2'} = \frac{s_3}{s_3'} = k \text{ then } \frac{Area\,of\,\Delta_1}{Area\,of\,\Delta_2} = k^2$$

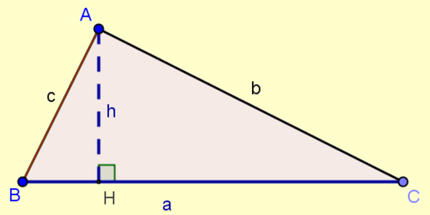
Metric relations in Similar Right Triangles

The right triangle is an amazingly rich geometric structure, with a multitude of relations between its elements.

Goal: Metric relations allow us to find missing measures in similar right triangles.

Height- Hypotenuse relation:

✓ From the two similar triangles *ABH* & *ABC*, we can write:

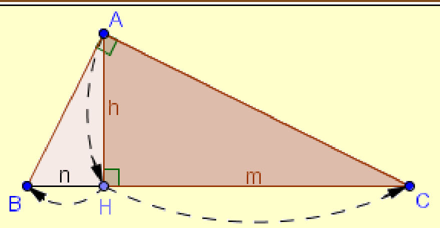
Formal way		Informal way
$\begin{array}{c} ABC \\ HBA \end{array} \Big\vert \frac{AB}{HB} = \frac{AC}{HA} = \frac{BC}{BA}$ (1, 2, 3) $\Rightarrow BC\times AH = AC\times B$	$BC\times AH = AB\times AC$	
Rule : $a\times h = b\times c.$	**Rule :** hyp.height = $leg_1 . leg_2$	

Geometric Means (Mean Proportional):

 BEWARE: The product of means equals the product of extremes.	$\frac{extreme}{mean} = \frac{mean}{extreme}$	In a mean proportional problem, "means" are the same values $\frac{a}{\square} = \frac{\square}{b}$  

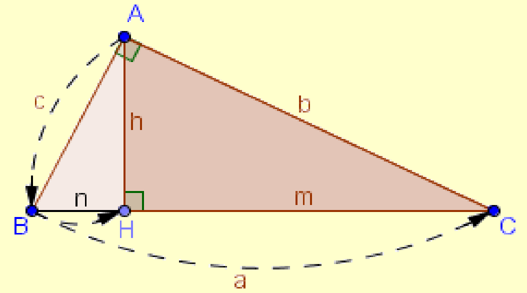
1- The altitude relative to the hypotenuse of a right triangle is the geometric mean (mean proportional) of the two segments of the hypotenuse.

✓ From the two similar triangles *ABH* & *ACH*, we can write:

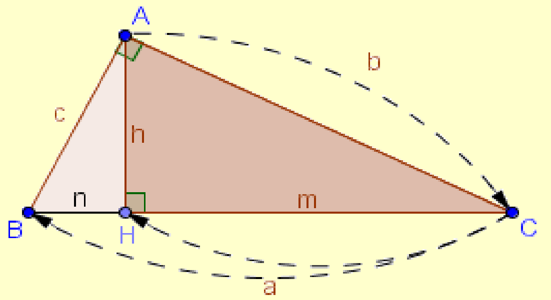
Formal way		Informal way
$\begin{array}{c} ABH \\ CAH \end{array} \Big\vert \frac{AB}{CA} = \frac{AH}{CH} = \frac{BH}{AH}$ (1, 2, 3) $\Rightarrow AH^2 = CH\times BH$	$\frac{BH}{AH} = \frac{AH}{CH}$	
Rule : $h^2 = m\times n.$	Altitude Rule : $\frac{\text{part of hyp}}{\text{altitude}} = \frac{\text{altitude}}{\text{other part of hyp}}$	

2- The length of a leg of a right triangle is the geometric mean of the lengths of the hypotenuse and the projection of this leg on the hypotenuse.

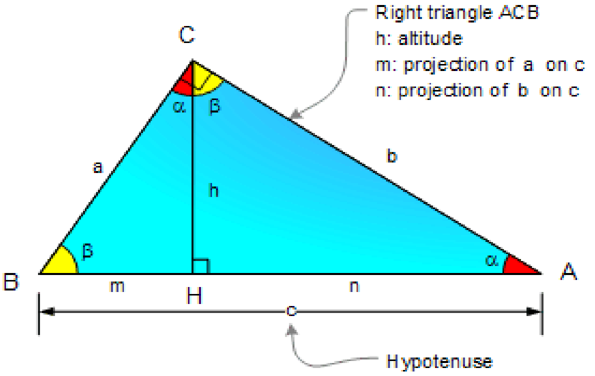
i- From the two similar triangles *ABH* & *ABC*, we can write:

Formal way		Informal way
$\begin{array}{l\|l} ABH & \frac{AB}{CB} = \frac{AH}{CA} = \frac{BH}{BA} \Rightarrow AB^2 = BH \times BC. \\ CBA & \ \ 1 \qquad\ 2 \qquad\ 3 \end{array}$		$\frac{BC}{AB} = \frac{AB}{BH}$
$\mathfrak{Rule}: c^2 = n \times a.$		*Leg Rule*: $\frac{\text{hypotenuse}}{\text{leg}_1} = \frac{\text{leg}_1}{\text{projection}}$ OR $\text{leg}_1^{\ 2} = \text{hypotenuse} \times \text{projection}.$

ii- From the two similar triangles *ACH* & *ABC*, we can write:

Formal way		Informal way
$\begin{array}{l\|l} ACH & \frac{AC}{BC} = \frac{AH}{BA} = \frac{CH}{CA} \Rightarrow AC^2 = CH \times BC. \\ BCA & \ \ 1 \qquad\ 2 \qquad\ 3 \end{array}$		$\frac{BC}{AC} = \frac{AC}{CH}$
$\mathfrak{Rule}: b^2 = m \times a.$		*Leg Rule*: $\frac{\text{hypotenuse}}{\text{leg}_2} = \frac{\text{leg}_2}{\text{projection}}$ OR $\text{leg}_2^{\ 2} = \text{hypotenuse} \times \text{projection}.$

The Magic of Similar Right Triangles All in One



To Prove:

1. $a^2 = c \cdot m,$ and $b^2 = c \cdot n$
2. $a^2 + b^2 = c^2$ (Pythagorean theorem)
3. $h^2 = m \cdot n$
4. $a \cdot b = c \cdot h$
5. $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{h^2}$

✓ A proof of Pythagorean theorem:

In $\Delta\ ABC$ we have:

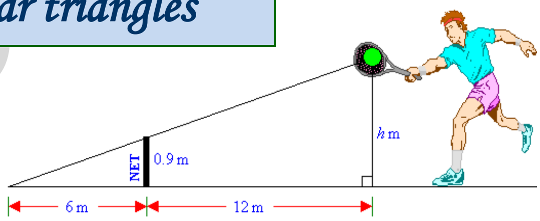
$$a^2 = m.c \quad ....(1) \quad \text{(Geomtric mean)}$$
$$b^2 = n.c \quad ....(2) \quad \text{(Geomtric mean)}$$

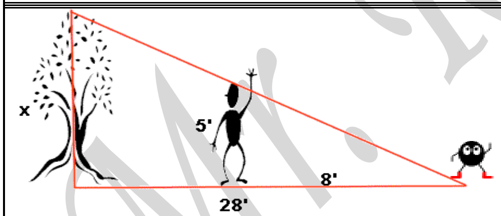
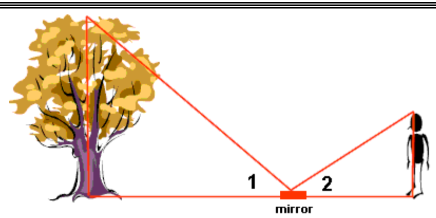
Add (1) & (2) to get:

$$a^2 + b^2 = n.c + m.c$$
$$a^2 + b^2 = c.(n+m)$$
$$but, c = m + n \qquad (B, H\ \&\ C \text{ are collinear})$$
$$Thus,\ c^2 = a^2 + b^2$$

Real life problems involving similar triangles

Find the height, h, in the following diagram at which the tennis ball must be hit so that it will just pass over the net and land 6 m away from the base of the net.



To find the height, h, of a tree or any tall object we use similar triangle techniques:	
1st – Technique	*2nd – Technique*
	
1- Allow your shadow and the shadow of the tall object to coincide. 2- Measure your height or any reference. 3- Measure distance between you and point of coincidence. 4- Measure distance between object and point of coincidence. 5- Use similar triangles to find ratio of measurements.	1- Set a mirror between you and the tall object to measure. 2- Move back till you can see the tip of the tall object in the mirror. 3- Measure your height or any reference. 4- Measure distance between you and the mirror. 5- Measure distance between object and the mirror. 6- Use similar triangles to find ratio of measurements.