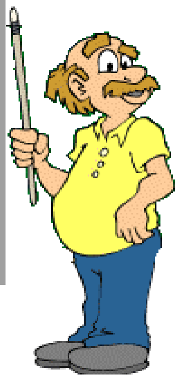


You have probably played with those solid wooden Japanese puzzles with interlocking pieces. When you start, it's the shape of a cube or a ball or perhaps an animal. Then you search around for the key piece that you can remove. Once you've pulled that one out, you can remove the remaining pieces one by one. Indeed once you have that first piece removed; it's relatively clear how to take the puzzle apart.

Taking the puzzle apart that's what you have learned to do when you learned to take derivatives. But in this section you will be starting with a pile of pieces lying on the table and you will have to figure out how to put them back together. Each problem of finding the indefinite integral of a function will be a new adventure. You will have to search around for the fit. And you will become efficient at searching out how the pieces fit only if you practice at it. It will be frustrating at first, but if you keep at it you'll get the hang of it.

The good news is that you will not be going off on this adventure unarmed. What you will learn in this section is a box of tools, each of which might be applicable to an integration problem. Sometimes you will have to apply more than one tool, and finding the order in which you apply them will be part of the puzzle. But before you can do that, you must first learn how to use each tool.

Calculus is a branch of mathematics concerned with such concepts as the rate of change of one variable quantity with respect to another. One of the early accomplishments of calculus was predicting the future position of a moving body from one of its known locations and a formula for its velocity function.

Integral calculus:

Integral calculus involves the inverse process of finding the derivative of a function, that is , it is the process of finding the function itself when its derivative is known. For instance, finding the equation of a curve if the slope of the tangent is known at an arbitrary point and finding distance in terms of time if the velocity (or acceleration) is known. Integral calculus can also find the areas of plane and curved surfaces, volumes of solids of revolution, centroids, moments of inertia, and total mass and total force.

Definition

A function $F(x)$ is an ***antiderivative*** of a function $f(x)$ if

$$F'(x) = f(x)$$

for all x in the domain of f. The set of all antiderivatives of f is the ***indefinite integral*** of f with respect to x, denoted by $\int f(x)dx$.

The symbol $\int$ is an ***integral sign***. The function f is the ***integrand*** of the integral and x is the ***variable of integration***.

$$\textit{Thus, } \int f(x)dx = F(x) + C$$

Integration of a Constant Function:

Suppose that a wire costs 3 cents per centimeter. This constant rate can be expressed as the function $y = f(x) = 3$ for all values of x, where x is the length of wire.

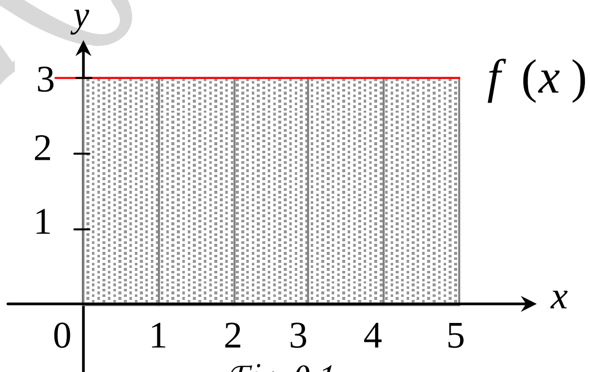


Fig- 0.1.

The area from 0 to 1 represents the 3 cents that the first centimeter of wire costs. All of the area under the “curve” from 0 to 5 represents the total, that the first five centimeters of wire cost, 15 cents. Adding the area under the curve is represented by the integral symbol ∫. The integral of the function y = 3 is 3x. The total cost is 5 cents.

Using integral calculus: The area of the shaded part is: $\int_0^5 3dx = 3x\Big|_0^5 = 3(5-0) = 15cent$.

Table 0.1 Integral formulas:

Basic integral formulas	
$1.\int x^{n}dx=\frac{x^{n+1}}{n+1}+C, s.t\ n\neq -1.$	$11.\int \frac{1}{x^{2}}dx=-\frac{1}{x}+C$
$2.\int \sin u du=-\frac{\cos u}{u'}+C.$	$12.\int \sinh u du=\frac{\cosh u}{u'}+C.$
$3.\int \cos u du=\frac{\sin u}{u'}+C.$	$13.\int \cosh u du=\frac{\sinh u}{u'}+C.$
$4.\int \sec^{2}x dx=\tan x+C.$	$14.\int \operatorname{sech}^{2}u du=\tanh u+C.$
$5.\int \csc^{2}x dx=-\cot x+C.$	$15.\int \operatorname{csch}^{2}u dx=-\frac{\cot u}{u'}+C.$
$6.\int \sec x \tan x dx=\sec x+C.$	$16.\int \operatorname{sech}u \tanh u du=-\operatorname{sech}u+C.$
$7.\int \csc x \cot x dx=-\csc x+C.$	$17.\int \operatorname{csch}u \coth u du=-\operatorname{csch}u+C.$
$8.\int \frac{dx}{x}=\ln\lvert x\rvert+C.$	$18.\int \frac{du}{a^{2}+u^{2}}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+C.$
$9.\int e^{ax}dx=\frac{e^{ax}}{a}+C.$	$19.\int \frac{du}{\sqrt{a^{2}-u^{2}}}=\sin^{-1}\left(\frac{u}{a}\right)+C.\quad s.t\ u^{2}<a^{2}$
$10.\int \frac{1}{\sqrt{x}}dx=2\sqrt{x}+C.$	$20.\int \frac{du}{u\sqrt{u^{2}-a^{2}}}=\frac{1}{a}\sec^{-1}\left\lvert\frac{u}{a}\right\rvert+C.\quad s.t\ u^{2}>a^{2}$

Techniques of integration:

1. Term by term integration:

Form: $\int f(x)\pm g(x)\pm q(x)\pm ...dx$.

In this method follow the steps:

Step-1: Separate each function with a different integral sign.

$\int f(x)dx \mp \int g(x)dx \mp \int q(x)dx \mp ...$

Step-2: Integrate with respect to x each of the integrals alone.

$F(x)+C_1\pm G(x)+C_2\pm Q(x)+C_3\pm ...$

Step-3: Add result and unify the constants of integrations with one constant only.

$F(x)\pm G(x)\pm Q(x)\pm ...+C$.

Ex-1: Integrate the function: $f(x)=3x^{2}-2x+1$.

Solution: $\int(3x^{2}-2x+1)dx=\int 3x^{2}dx-\int 2x dx+\int dx$.

$$=\frac{3x^{3}}{3}+C_1-\frac{2x^{2}}{2}+C_2+x+C_3.$$

$$=x^{3}-x^{2}+x+C.$$

2. Substitution method:

Form: $\int f\,\left(g(x)\right).g'(x)dx$.

In this method follow the steps:

Step-1: Substitute $u = g(x)\, and\ du = g'(x)dx\ to\, obtain\ \int f\,(u)du$.

Step-2: Integrate with respect to u.

Step-3: Replace u by $g(x)$ in the result.

Ex-2: Integrate:

$$I_1 = \int 2x(x^2-1)^2 dx$$

$$I_2 = \int 3x^2 \sin x^3 dx$$

$$I_3 = \int \sin t \cos^2 t dt$$

Solution: Step-1: $I_1 = \int \underbrace{(x^2-1)^2}_{u} \underbrace{2xdx}_{du}$

Let $u = x^2 - 1 \Rightarrow du = 2xdx$

$\Rightarrow I_1 = \int 2x(x^2-1)^2 dx = \int u^2 du$.

Step-2: $I_1 = \int u^2 du = \dfrac{u^3}{3} + C$.

Step-3: $Now, replace$

$u = x^2 - 1 in\, I_1 to\ get$

$$\therefore I_1 = \frac{\left(x^2-1\right)^3}{3} + C$$

Solution: Step-1: $I_2 = \int \sin \underbrace{x^3}_{v} \underbrace{3x^2 dx}_{dv}$

Let $v = x^3 \Rightarrow dv = 3x^2 dx$

$\Rightarrow I_2 = \int 3x^2 \sin x^3 dx = \int \sin v\, dv$

Step-2: $I_2 = \int \sin v dv = -\cos v + C$.

Step-3: $Now, replace$

$v = x^3\, in\, I_2\, to\ get$

$\therefore I_2 = -\cos x^3 + C$.

Solution: $I_3 = \int \underbrace{\cos^2 t}_{w} \underbrace{\sin t dt}_{dw}$

Let $w = \cos t \Rightarrow dw = -\sin t dt$

$\Rightarrow I_3 = \int \sin t \cos^2 t dt = -\int w^2 dw$

$I_3 = -\int w^2 dw = -\dfrac{w^3}{3} + C$ (Substitute $w = \cos t$ in I_3 to get)

$$\therefore I_3 = -\frac{\cos^3 t}{3} + C.$$

3. Integration by parts:

If substitution does not work try integration by parts
Form: $\int f(x)g(x)dx$.

Step-1: Match the given integral with the form $\int udv$

Step-2: Choose u to be $f(x)$ such that the differentiation of $f(x)$ eventually terminates.
Step-3: Choose dv to be $g(x)$ including dx so that differentiation of $g(x)$ doesn't end.
Step-4: Apply the formula $\int udv = uv - \int vdu$ which gives a simpler integral.

Ex-3: Find $\int x \sin x dx$

Solution: $\int x \sin x dx$

$Let\ u = x$ (Since x terminates after two derivatives)

$Let\ dv = \sin x dx$ (Since $\sin x dx$ does not terminate)

$\Rightarrow \int x \sin x dx = \int udv$

$\Rightarrow du = 1.dx$

$and\ dv = \sin x dx$ (integrate both sides)

$\Rightarrow \int dv = \int \sin x dx$

$\Rightarrow v = -\cos x$

Now, substitute obtained results in formula $\int udv = uv - \int vdu$ to get

$\int x \sin x dx = x(-\cos x) - \int (-\cos x)dx$

$\therefore \int x \sin x dx = -x \cos x + \sin x + C$.

4. Tabular integration:

If integration by parts proved to be difficult then use this method.
Form: $\int f(x)g(x)dx$.

Ex-4: Find $\int x^2 e^x dx$ use tabular method

Solution: $Let\ f(x) = x^2\ and\ g(x) = e^x$ then,

f(x) and its derivatives		g(x) and its integrals
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x

Add the products of the functions connected by arrows keeping signs, to get

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C.$$

5. <u>Reduce improper fraction</u>:

A fraction is improper if degree of numerator is greater than or equal to that of denominator.

Form: $\int \frac{P(x)}{Q(x)}dx \; where, P(x) \neq Q'(x) \; and \; vice versa.$

In this method follow the steps:

Step-1: divide fraction to obtain a quotient plus a remainder (proper fraction)

Step-2: integrate using term by term technique.

Ex-5: Evaluate $A = \int \frac{3x^2 - 7x}{3x+2}dx$.

Solution: $\frac{3x^2 - 7x}{3x+2} = x - 3 + \frac{6}{3x+2}$.

$$\begin{array}{r} x - 3 \\ 3x+2 \,\big)\, \overline{3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +6 \end{array}$$

$\Rightarrow A = \int \frac{3x^2 - 7x}{3x+2}dx = \int \left(x - 3 + \frac{6}{3x+2} \right) dx$

$= \int x dx - \int 3 dx + \int \frac{6}{3x+2}dx = \frac{x^2}{2} - 3x + \int \frac{6}{3x+2}dx$

Let $u = 3x + 2 \Rightarrow du = 3dx$

$\Rightarrow \int \frac{6}{3x+2}dx = \int \frac{2}{u}du = 2\ln|u| = 2\ln|3x+2|$

$\therefore A = \frac{x^2}{2} - 3x + 2\ln|3x+2| + C.$

6. <u>Reduce partial fraction</u>:

Forms: *Proper fraction* *Decomposition*

$$\frac{numerator}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

$$\frac{numerator}{(ax+b)^n} = \frac{A}{(ax+b)^1} + \frac{B}{(ax+b)^2} + \dots + \frac{N}{(ax+b)^{n-1}} + \frac{R}{(ax+b)^n}.$$

$$\frac{numerator}{(x^2+p)(x+q)^2} = \frac{Ax+B}{x^2+p} + \frac{C}{(x+q)} + \frac{D}{(x+q)^2}.$$

Ex-6: Find $\int \frac{5x-3}{(x+1)(x-3)}dx$.

Solution: $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x-3)}$

$\Rightarrow (5x-3) = A(x-3) + B(x+1) = (A+B)x = 3A + B$ ...... (1)

$eq.(1)\; gives \quad \begin{cases} A + B = 5 \\ -3A + B = -3 \end{cases} \quad solve\; system\; to\; get\; A = 2 \; and \; B = 3.$

$\Rightarrow \int \frac{5x-3}{(x+1)(x-3)}dx = \int \frac{2}{(x+1)}dx + \int \frac{3}{(x-3)}dx$

$= 2\ln|x+1| + 3\ln|x-3| + C.$

Ex-7: Evaluate $\int \frac{6x+7}{(x+2)^2}dx$.

Solution: $\frac{6x+7}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$ .....(1)

$eq.(1)\, gives : 6x+7 = A(x+2) + B = Ax + 2A + B.$

Match the coefficients of like terms to get:

$A = 6.$

$and, 7 = 2A + B$

$\Rightarrow B = -5.$

$$\Rightarrow \int \frac{6x+7}{(x+2)^2}dx = \int \frac{6}{(x+2)}dx - \int \frac{5}{(x+2)^2}dx.$$

$$= 6\ln|x+2| + \frac{5}{x+2} + C.$$

Ex-8: Compute $\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$.

Solution: $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{(x-1)} + \frac{D}{(x-1)^2}$ ..... (1)

$eq.(1)\, gives :$

$$-2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1)$$

$$= (A+C)x^3 + (-2A+B-C+D)x^2 + (A-2B+C)x + (B-C+D).$$

Equate coefficients of like terms to get:

$coeff\ of\ x^3:\quad 0 = A + C$

$coeff\ of\ x^2:\quad 0 = -2A + B - C + D$

$coeff\ of\ x^1:\quad -2 = A - 2B + C$

$coeff\ of\ x^0:\quad 4 = B - C + D$

Solve the above equations simultaneously to give:

$A = 2, B = 1, C = -2, and\ D = 1$

$$\Rightarrow \int \frac{-2x+4}{(x^2+1)(x-1)^2}dx = \int \frac{2x+1}{x^2+1}dx - \int \frac{2}{(x-1)}dx + \int \frac{1}{(x-1)^2}dx.$$

$$= \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx - \int \frac{2}{x-1}dx + \int \frac{1}{(x-1)^2}dx$$

$$= \ln(x^2+1) + \tan^{-1}x - 2\ln|x-1| - \frac{1}{x-1} + C.$$

Exercises 0.1

I- Evaluating integrals using table 0.1:

1. $\int (2x-1)dx$.
2. $\int \left(3x^2 - \dfrac{1}{x^2}\right)dx$.
3. $\int \left(\dfrac{\sqrt{z}}{2} + \dfrac{2}{\sqrt{z}}\right)dz$.
4. $\int \dfrac{2}{t-1}dt$.
5. $\int 6(\cos 2\theta + \sin 3\theta)d\theta$.
6. $\int \cos x(\tan x + \sec x)dx$
7. $\int \dfrac{\csc t}{\csc t - \sin t}dt$.
8. $\int 4\sin^2 z\,dz$.
9. $\int (2+\tan^2\theta)d\theta$.
   (Hint: $1+\tan^2\theta = \sec^2\theta$)
10. $\int \cot^2\theta\, d\theta$.

II- Verify the following integral formulas by differentiation:

1. $\int (3x-1)^2 dx = \dfrac{(3x-1)^3}{9} + C$.
2. $\int (2t-3)^{-2} dt = -\dfrac{(2t-3)^{-1}}{2} + C$.
3. $\int \sec^2\left(\dfrac{2z+5}{7}\right)dz = \dfrac{7}{2}\tan\left(\dfrac{2z+5}{7}\right) + C$.
4. $\int \csc^2(5t-4)dt = -\dfrac{1}{5}\cot(5t-4)$.
5. $\int \dfrac{1}{(x+1)^2}dx = -\dfrac{1}{x+1} + C$.
6. $\int \dfrac{1}{(x+1)^2}dx = \dfrac{x}{x+1} + C$.
7. $\int 3e^{3x}dx = e^{3x} + C$.
8. $\int \dfrac{2x}{x^2-3}dx = \ln\left|x^2-3\right| + C$.

III- Check the correctness of the following. Support your choice briefly.

1. $a)\ \int \sqrt{2x-5}\,dx = \sqrt{x^2-4x} + C$.
   $b)\ \int \sqrt{2x-5}\,dx = \sqrt[3]{(2x-5)^2} + C$.
   $c)\ \int \sqrt{2x-5}\,dx = \dfrac{1}{3}\sqrt{(2x-5)^3} + C$.
2. $a)\ \int (7t-3)^2 dt = \dfrac{(7t-3)^3}{3} + C$.
   $b)\ \int 3(7t-3)^2 dt = (7t-3)^2 + C$.
   $c)\ \int 21(7t-3)^2 dt = (7t-3)^3 + C$.

IV- Compute the following integrals:

1. $\int 2x(x^2-3)^3 dx$.
2. $\int \frac{x^2-1}{x^3-3x+5}dx$.
3. $\int \frac{1+x}{7-2x-x^2}dx$.
4. $\int (3x-2)e^{x^3-2x+1}dx$.
5. $\int x\sqrt[3]{x^2-4}dx$.
6. $\int e^{2x}(3-e^{2x})^3 dx$.
7. $\int \frac{e^x}{1-e^x}dx$.
8. $\int \frac{\ln|x-2|}{x-2}dx$.
9. $\int \frac{(\ln x)^3}{x}dx$.
10. $\int t^3(1+t^4)^3 dt$.
11. $\int \csc^2 2\theta \cot 2\theta d\theta$.
12. $\int \sin^5 \frac{x}{3} \cos\frac{x}{3}dx$.
13. $\int \sec(u+\frac{\pi}{2})\tan(u+\frac{\pi}{2})du$.
14. $\int \sqrt{x}\sin^2(x^{3/2}-5)dx$.

V- Evaluate the integrals:

1. $\int \frac{dx}{1-x^2}$.
2. $\int \frac{dx}{x^2+2x}$.
3. $\int \frac{3x^2-7x}{3x+2}dx$.
4. $\int \frac{5x-3}{x^2-2x-3}dx$.
5. $\int \frac{x+4}{x^2+5x-6}dx$.
6. $\int \frac{dt}{t^3+t^2-2t}$.
7. $\int \frac{dx}{(x+1)(x^2+1)}$.
8. $\int \frac{2s+2}{(s^2+1)(s-1)^3}ds$.
9. $\int \frac{2s^3-2s^2+1}{s^2-s}ds$.
10. $\int \frac{9r^2-3r+1}{r^3-r^2}dr$.
11. $\int \frac{t^4+t^2-1}{t^3+t}dt$.

VI- Find the value of the following integrals:

1. $\int xe^{3x}dx$.
2. $\int x^2\cos x dx$.
3. $\int x^2 \ln x dx$.
4. $\int x^3 e^x dx$.
5. $\int x^3 \sin^2 x dx$.

VII- Economic Problems:

1. Find the cost function $C(x)$ and the cost of producing twenty units of a certain item, if the marginal cost of producing x units is given by
$$C'(x) = 4x^3 - 3x^2.$$
where the fixed cost of production is 2 000$.

2. Find the revenue function $R(x)$ when the marginal revenue is
$$R'(x) = 300 - 0.3x$$
and no revenue results at zero production level. What is the revenue at a production level of 1 000 units?

3. The marginal price $p'(x)$ at x units per month demand for a given model sailboat is given by
$$p'(x) = -500e^{-0.05x}$$
Find the price – demand equation if at a price $17 788 each the demand is 5 boats per month.

VIII-Life Science Problems:

1. The rate of change of an average person's weight with respect to their height h (in cm) is given by
$$\frac{dW}{dh} = 0.0015h^2$$
Find $W(h)$ if $W(172) = 72$ kilograms. Find the average weight for a person who is 1m and 82cm tall.

2. The blood pressure in the aorta (largest artery in a human body) under certain restrictions changes between beats with respect to time t according to
$$\frac{dP}{dt} = -aP \qquad P(0) = P_0$$
where a is a constant. Find $P = P(t)$ that satisfies both conditions.

3. A single injection of a drug is administered to a patient. The amount Q in the body then decreases at a rate proportional to the amount present, and for this particular drug the rate is 4% per hour. Thus,
$$\frac{dQ}{dt} = -0.04Q \qquad Q(0) = Q_0$$
where t is the time in hours. If the initial injection is 3 millimeter, find $Q = Q(t)$ that satisfies both conditions. How many millimeters of the drug are still in the patient's body after 10 hours?

Double Integration — *Chapter-1*

In this chapter we will deal with functions of two or three variables, which will be similar to those with single valued functions only more general, moreover we will be able to evaluate plane areas, volumes, surface areas, moments and center of mass.

1.1 Double Integrals over rectangular regions:

Consider the two variable function $f(x,y)$ to be defined on the rectangular region R given by

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

Divide region R into n small rectangles each of area $\Delta A = \Delta x\, \Delta y$. Numerate these areas $\Delta A_1, \Delta A_2, ..., \Delta A_n$, choose a point (x_k, y_k) in each rectangle ΔA_k and form sum

$$s_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k. \qquad (1)$$

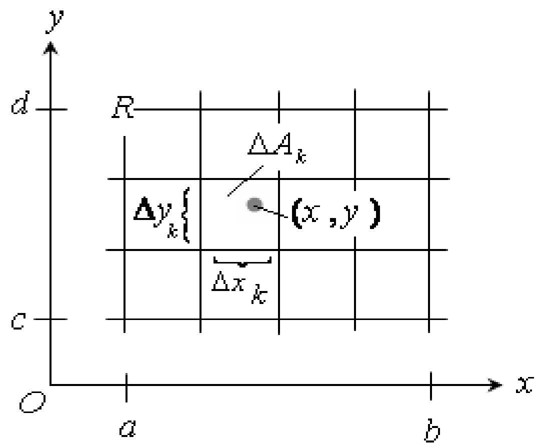


Fig-1.1.

If *f* is continuous throughout *R*, then, as we refine the web to allow Δx and Δy to tend to zero, the sum in equation (1) will approach a limit called the ***double integral*** of *f* over the given region *R*.

That is to say, $\lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \iint_R f(x,y)dA.$

Then we have the notation: $\iint_R f(x,y)dA.$

1.2 Properties of double integrals:

Double integrals of continuous functions have algebraic properties that are useful in computations and in various applications.

$a. \iint_R \lambda f(x,y)dA = \lambda \iint_R f(x,y)dA.$ (s.t λ is any arbitrary number)

$b. \iint_R (f(x,y) \pm g(x,y))dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA.$

$c. \iint_R f(x,y)dA \ge 0 \text{ if } f(x,y) \ge 0 \text{ in } R.$

$d. \iint_R f(x,y)dA \ge \iint_R g(x,y)dA \text{ if } f(x,y) \ge g(x,y) \text{ in } R.$

$e. \iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$ (s.t $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \varnothing$)

1.3 Rules of calculation of double integrals:

To calculate a double integral in a rather simple way we will use the theorem presented and proved in 1907 by Guido Fubini, which says that the double integral of any continuous functions over a rectangular region can be calculated as an iterated integral in either order of integration.

> Theorem 1
>
> Fubini's Theorem (First Form)
>
> If $f(x,y)$ is continuous on the rectangular region $R: a \le x \le b, c \le y \le d$,
>
> then
>
> $$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Ex-1: Calculate $\iint_R \mathrm{f(x,y)dA}$ for

$$f(x,y) = 1 + 6xy^2 \text{ and } R: 0 \le x \le 2,\ -1 \le y \le 1.$$

Solution: By Fubini's theorem,

$$\iint_R f(x,y)\,dA = \int_{-1}^{1}\int_{0}^{2}(1+6xy^2)\,dx\,dy$$

$$= \int_{-1}^{1}\left[x + 3x^2y^2\right]_{x_1=0}^{x_2=2} dy$$

$$= \int_{-1}^{1}(2+12y^2)\,dy$$

$$= \left[2y + 4y^3\right]_{y_1=-1}^{y_2=1} = 12.$$

Reversing the order of integration

$$\iint_R f(x,y)\,dA = \int_{0}^{2}\int_{-1}^{1}(1+6xy^2)\,dy\,dx$$

$$= \int_{0}^{2}\left[y + 2xy^3\right]_{y_1=-1}^{y_2=1} dx$$

$$= \int_{0}^{2}(2+4x)\,dx$$

$$= \left[2x + 2x^2\right]_{x_1=0}^{x_2=2} = 12.$$

Therefore, reversing the order of integration gives the same answer.

1.4 Double Integration over Bounded Nonrectangular Regions

To define the double integral of a function $f(x,y)$ over a bounded nonrectangular region,

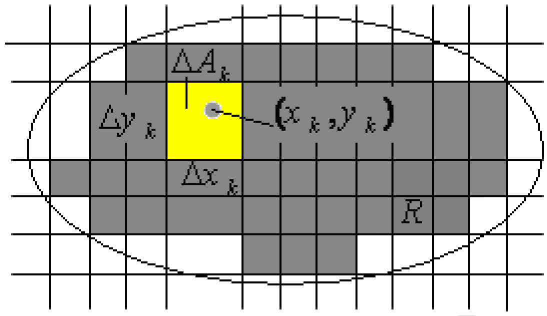


Fig-1.2.

We again imagine that R is covered by a rectangular grid, but we will include in the partial sum only the pieces of area $\Delta A = \Delta x\,\Delta y$ that lie entirely within the shaded region. We number the pieces in some order, choose an arbitrary point (x_k, y_k) in each ΔA_k, and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k .$$

Note that in the above sum the areas ΔA_k may not cover the entire region *R*. But as the mesh becomes increasingly fine and the number of terms in S_n increases, more and more of *R* is included.

If *f* is continuous and the boundary of *R* is made from the graphs of a finite number of smooth continuous functions of *x* and/or continuous functions of *y*, then the sum S_n will have a limit as the norms of the partitions that define the rectangular grid independently approach zero.

We call the limit the double integral of *f* over *R*:

$$\iint f(x,y)\,dA = \lim_{\Delta A \to 0} \sum f(x_k, y_k)\Delta A_k .$$

Theorem 2

Fubini's theorem (Stronger Form)

Let $f(x,y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b, g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R \text{f(x,y)dA} = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

2. If R is defined by $c \le y \le d, h_1(x) \le x \le h_2(x)$, with h_1 and h_2 continuous on $[c,d]$, then

$$\iint_R \text{f(x,y)dA} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

1.5 Double Integration as Volume

If $f(x,y)$ is a positive and continuous function, then we may interpret the double integral of f over a curved base R as the volume of the solid bounded between R in the xy-plane and whose top is the surface $z = f(x,y)$.

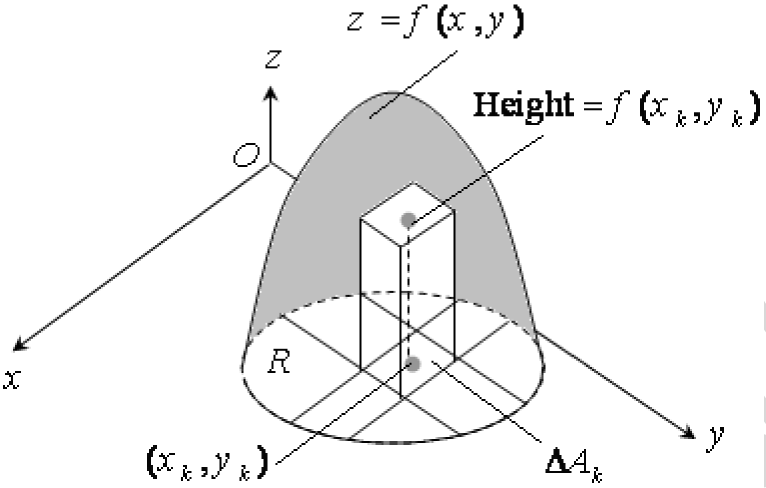


Fig-1.3.

Therefore,

$$\text{Volume} = \lim \sum f(x_k, y_k)\Delta A_k = \iint_R f(x,y)\,dA.$$

Ex-2: Find the volume of the prism whose base is a triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x,y) = 3 - x - y.$$

Solution:

1st step: sketch required region

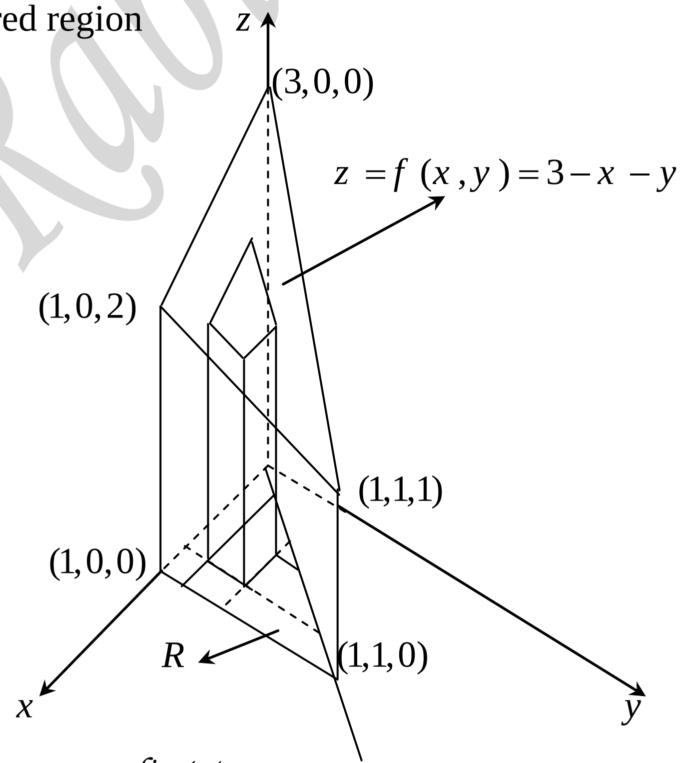
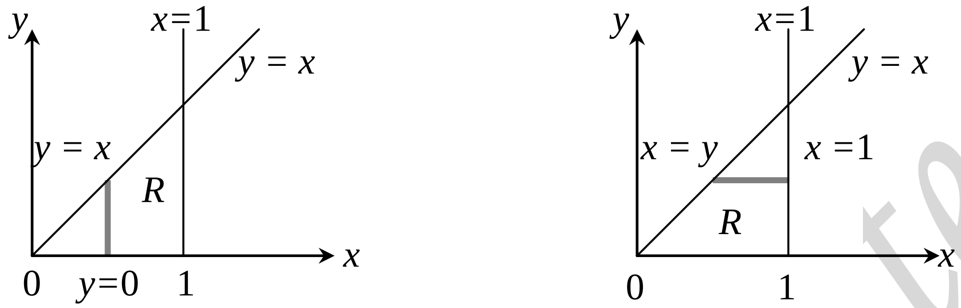


fig-1.4.

2^{nd} step: simplifying the region of integration

The above sketch may be a bit difficult to deal with, so to find the bounds of integration for the given solid in a rather simpler way we will project the prism on *xy*-plane then determine the limits of integration with respect to *x* and *y*.



a *fig-1.5.* *b*

3^{rd} step: use Fubini's theorem

If we integrate with respect to *y* then with respect to *x* by using (fig-1.5a), then you get the integral:

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x,y)\,dy\,dx.$$

To integrate first with respect to *y*, we integrate along a vertical line through *R* and then integrate from left to right to include all vertical lines in *R*.

If we integrate with respect to *x* then with respect to *y* by using (fig-1.5a), then you get the integral:

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} f(x,y)\,dx\,dy.$$

To integrate first with respect to *x*, we integrate along a horizontal line through *R* and then integrate from bottom to top to include all horizontal lines in *R*.

4^{th} step: computation of the volume

$$V=\int_0^1\int_y^1(3-x-y)\,dx\,dy$$

$$=\int_0^1\left[3x-\frac{x^2}{2}-xy\right]_{x=y}^{x=1}dy$$

$$=\int_0^1\left(\frac{5}{2}-4y+\frac{3}{2}y^2\right)dy$$

$$=\left[\frac{5}{2}y-2y^2+\frac{y^3}{2}\right]_{y=0}^{y=1}$$

$$=1.$$

If you try the reversed order you will get the same answer.

Exercises 1.1-5.

I- Evaluate the following iterated integrals:

1. $\int_0^1\int_0^2 (x+y)\,dy\,dx$.
2. $\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx$
3. $\int_1^2\int_0^4 (x^2-2y^2+1)\,dx\,dy$.
4. $\int_0^1\int_y^{2y} (1+2x^2+2y^2)\,dx\,dy$.
5. $\int_0^1\int_0^{\sqrt{1-y^2}} (x+y)\,dx\,dy$.
6. $\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy$.
7. $\int_0^2\int_0^{\sqrt{4-y^2}} \dfrac{2}{\sqrt{4-y^2}}\,dx\,dy$.
8. $\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta$.
9. $\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\,dr\,d\theta$.
10. $\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta$.

II- Sketch the region of integration and evaluate the following integrals:

1. $\int_0^3\int_0^2 (4-y^2)\,dy\,dx$.
2. $\int_\pi^{2\pi}\int_0^\pi (\sin x+\cos y)\,dx\,dy$.
3. $\int_0^\pi\int_0^x x\sin y\,dy\,dx$.
4. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y}\,dx\,dy$.
5. $\int_0^\pi\int_0^{\sin x} y\,dy\,dx$.
6. $\int_0^2\int_0^{4-y^2} y\,dx\,dy$.
7. $\int_{-2}^0\int_v^{-v} 4\,dp\,dv$. (the $pv-plane$)
8. $\int_0^1\int_0^{\sqrt{1-s^2}} 8t\,dt\,ds$. (the $st-plane$)

III- Integrate each of the given functions over its region:

1. $f(x,y)=x/y$ over the region in the first quadrant bounded by the lines $y=x$, $y=2x$, $x=1$, *and* $x=2$.
2. $f(s,t)=e^s\ln t$ over the region in the first quadrant of the *st*-plane that lies above the curve $s=\ln t$ from $t=0$ to $t=2$.
3. $f(x,y)=y\cos xy$ over the rectangular region $0\le x\le\pi, 0\le y\le 1$.

IV- Set up the integral for both orders of integrations, and use the more convenient order to evaluate the integral over the region *R*.

1. $\iint_R xy\,dA$ $\quad R:$ rectangle with vertices $(0,0),(0,5),(3,5),$ and $(3,0)$.
2. $\iint_R \sin x\sin y\,dA$ $\quad R:$ the rectangle $(-\pi,0),(\pi,0),(\pi,\pi/2),$ and $(-\pi,\pi/2)$.

V- Find the following volumes:

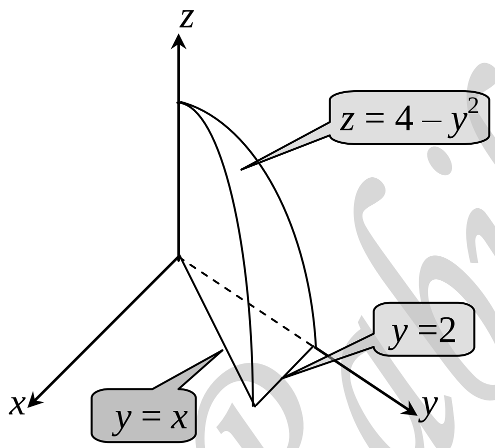
1. Volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $y = x, x = 0,$ and $x + y = 2$ in the $xy - plane$.
2. Volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3,$ and the parabolic cylinder $z = 4 - y^2$.

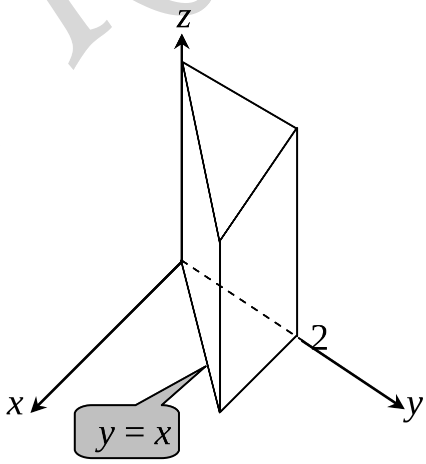
VI- A solid right non circular cylinder has its base *R* in the *xy*-plane and is bounded from above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

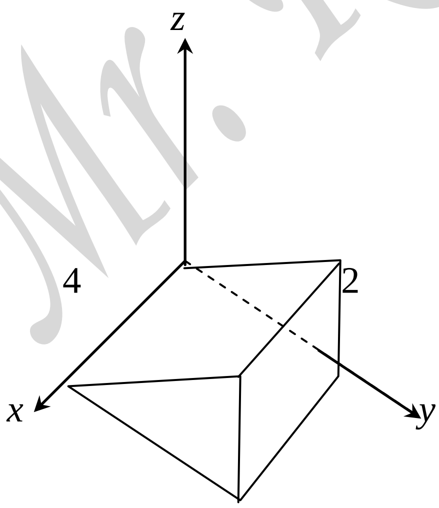
$$V = \int_0^1 \int_0^y (x^2 + y^2) dx dy + \int_1^2 \int_0^{2-y} (x^2 + y^2) dx dy .$$

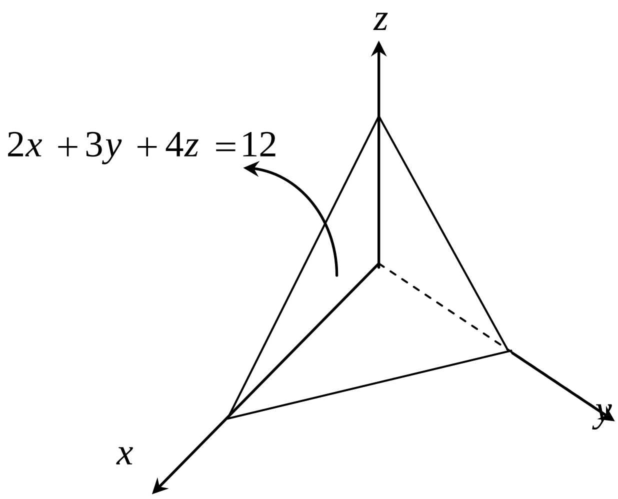
a. Sketch the region *R*, the base of the given solid.
b. Express the cylinder's volume as a single iterated integral with the order of integration reversed.
c. Evaluate the integral to find the volume.

VII- Use a double integral to find the volume of the specified solids:









1.6 Geometric and Physical Applications of Double Integrals

In this section we will benefit of the double integral to define and calculate the areas of bounded regions in a plane, average value, masses, moments, center of masses and moments of inertia.

A. Areas of bounded regions in a plane:

If the integrand $f(x,y)=1$ in a given double integral over a region *R*, then the partial sum of (eq-1) is reduced to

$$S_n=\sum_{k=1}^{n} f(x_k,y_k)\Delta A_k=\sum_{k=1}^{n} 1.\Delta A_k$$

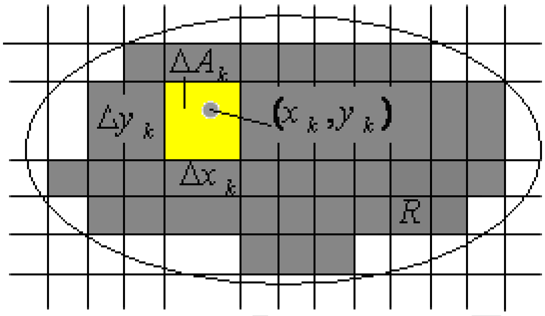


fig-1.6.

This approximation will be considered as the area of *R*. As Δx and Δy approach zero, the ΔA's will increasingly cover the region *R*.

Definition

The area of a closed, bounded plane region R is given by the formula

$$\text{Area}=\iint_R dA.$$

B. Average value:

Definition

The average value of a two variable function defined on a closed and bounded region is given by the formula

$$\text{Average value of } f \text{ over } R=\frac{1}{\text{area of } R}\iint_R f\, dA.$$

If *f* is the area density of a thin plate covering *R*, then the double integral of f over *R* divided by the area of *R* is the plate's average density in units of mass per unit area.

If $f(x,y)$ is the distance from the point (x,y) to a fixed point *P*, then the average value of *f* over *R* is the average distance of the points in *R* from *P*.

C. Mass:

> ***Definition***
>
> If ρ is a continuous density function on the lamina corresponding to a plane region *R*, then the mass *m* of the lamina is given by
>
> $$mass = \iint_R \rho(x, y)\, dA.$$

If the lamina corresponding to a region *R* (fig-1.7), has a constant density function ρ, then the mass of the lamina is given by

$$Mass = \iint_R \rho\, dA = \rho \iint_R dA.$$

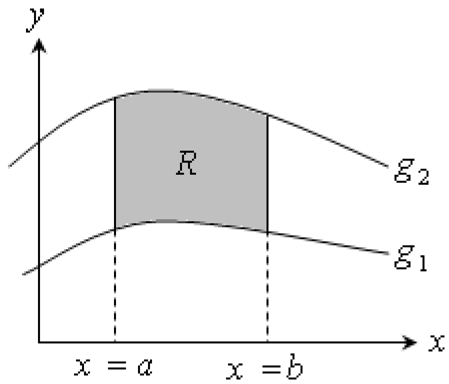


Fig-1.7. (lamina of constant density)

Note: Density is normally expressed as mass per unit volume. For a planar lamina, however, density is mass per unit surface area.

D. Moments, Moments of inertia and Center of Mass

For a partition Δ of a lamina of variable density corresponding to a plane region *R*, consider the *k*th rectangle R_k of area ΔA_k see fig-1.8. Suppose that the mass of R_k is concentrated at one of its interior points (x_k, y_k).The moment of mass of R_k with respect to *x*-axis is approximated by

$$(mass)(y_k) \approx \left[\rho(x_k, y_k)\Delta A_k\right](y_k).$$

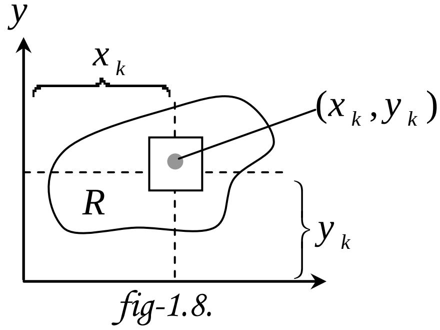


fig-1.8.

Similarly, the moment of mass with respect to the *y*-axis is approximated by

$$(mass)(x_k) \approx \left[\rho(x_k, y_k)\Delta A_k\right](x_k).$$

By forming Riemann sum of all such products and taking limits as norm of Δ approaches 0, we obtain the definitions of moments of mass with respect to *x*- and *y*-axes.

Ex-3: Find the mass of the triangular lamina with vertices $(0,0),(0,3), \text{and} (2,3),$ given that the density at (x,y) is $\rho(x,y) = 2x + y$.

Solution: Draw region of integration then indicate boundaries

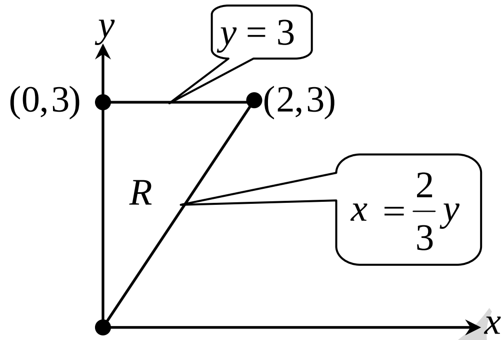


fig-1.9.

$$M = \iint_R (2x+y)\,dA = \int_0^3 \int_0^{2y/3} (2x+y)\,dx\,dy$$

$$= \int_0^3 \left[x^2 + xy\right]_0^{2y/3} dy = \frac{10}{9}\int_0^3 y^2\,dy = \frac{10}{9}.\frac{y^3}{3}\Bigg]_0^3 = 10.$$

Moments, Moments of inertia and Center of mass

Let ρ be a continuous density function on the planar lamina *R*. The moments of mass or 1st moments with respect to the x- and y-axes are

$$M_x = \iint_R y\,\rho(x,y)\,dA \qquad and \qquad M_y = \iint_R x\,\rho(x,y)\,dA.$$

If M is the mass of the lamina, then the center of mass is

$$\left.\begin{aligned} \bar{x} &= \frac{M_y}{M} \\ \bar{y} &= \frac{M_x}{M} \end{aligned}\right\} \Rightarrow \left(\bar{x}, \bar{y}\right).$$

If the density function is constant and equal to 1 then the center of mass of a plane region is called the centroid of the shape.

Moments of inertia (2nd moments)

$$\left.\begin{aligned} &\text{About the } x-axis \quad I_x = \iint_R y^2 \rho(x,y)\,dA \\ &\text{About the } y-axis \quad I_y = \iint_R x^2 \rho(x,y)\,dA \end{aligned}\right\} \Rightarrow About\ the\ origin\ \ I_o = I_x + I_y$$

$$\therefore I_o = \iint_R (x^2 + y^2)\rho(x,y)\,dA.$$

1.7 Change of variables in Double Integrals

1.7.1 Polar Coordinates

Double integral in rectangular coordinates can sometimes be difficult depending on the shape of the region and the type of the function, so we tend to change the coordinates of the region of integration from Cartesian to Polar. Especially in regions such as circles, cardioids, and petal curves, and for integrals that involve $x^2 + y^2$.

Exercises 1.6

I- Sketch the region of integration and compute the area

$$\int_{1}^{2}\int_{y}^{y^2} dx\,dy\,.$$