The Islamic Institution For Education & Teaching Al-Mahdi Schools	Answer Key	Mathematics Department
Class: Grade 11 (Scientific)	**Mid-Year Exam**	**2009-2010**

Q	Parts	Elements of answer	Notes
I	1)	$\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ So, $\cos^2\dfrac{\pi}{8} = \dfrac{1+\cos\left(\dfrac{\pi}{4}\right)}{2}$ So, $\cos^2\dfrac{\pi}{8} = \dfrac{1+\dfrac{\sqrt{2}}{2}}{2}$ so, $\cos^2\dfrac{\pi}{8} = \dfrac{2+\sqrt{2}}{4}$ so, $\left\|\cos\dfrac{\pi}{8}\right\| = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ but $\dfrac{\pi}{8} \in]0;\dfrac{\pi}{2}[$ thus, $\cos\dfrac{\pi}{8} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ A	
	2)	$(C): 2x^2+2y^2+4x-8y+4=0$ Then, $x^2+y^2+2x-4y+2=0$ Which is of the form: $x^2+2y^2-2ax-2by+c=0$ Where, $radius = \sqrt{a^2+b^2-c}$ and center $(a;b)$ Thus, $\Omega\left(\dfrac{-2}{2};\dfrac{4}{2}\right)$ & $r=\sqrt{3}$ D	`
	3)	$\lim\limits_{x\to-\infty}\dfrac{2x+3}{\sqrt{9x^2-1}} = \lim\limits_{x\to-\infty}\dfrac{x\left(2+\dfrac{3}{x}\right)}{\sqrt{x^2\left(9-\dfrac{1}{x^2}\right)}}$ $= \lim\limits_{x\to-\infty}\dfrac{2x}{\sqrt{9x^2}}$ $= \lim\limits_{x\to-\infty}\dfrac{2x}{3\|x\|}$ but $x<0$ Thus, $\lim\limits_{x\to-\infty}\dfrac{2x+3}{\sqrt{9x^2-1}} = -\dfrac{2}{3}$ C	
II	1 a)	$f(x)$ is of the form, $r+n$ Where, $r=\dfrac{a}{u}, so, r' = \dfrac{-au'}{u^2}$ and, $n=\sqrt{v}, so, n' = \dfrac{v'}{2\sqrt{v}}$ Thus, $f'(x) = \dfrac{-3(2x)}{(x^2+1)^2} + \dfrac{2x}{2\sqrt{x^2-1}}$	
	b)	$f(x)=\sin^2(x^2)$ is of the form, $\sin^n u$ Then, $(\sin^n u)' = nu'\cos u \cdot \sin^{n-1}(u)$ Thus, $f'(x) = 2(2x)\cos x^2 \sin x^2$	

	2		$f(x)=\sqrt{\dfrac{2x^2+3x-5}{x^2+x-2}}$ is defined *if &only if*, $\begin{cases}2x^2+3x-5\geq 0......(I)\\x^2+x-2>0..........(II)\end{cases}$ Using-I: $a+b+c=0$ so, $x_1=1\&\ x_2=\dfrac{-5}{2}$, but $a>0$ So, $(I)\geq 0$ for all $x\in]-\infty;\dfrac{-5}{2}[\cup]1;+\infty[$ Using-II: $a+b+c=0$ so, $x_1=1\&\ x_2=-2$, but $a>0$ So, $(II)\geq 0$ for all $x\in]-\infty;-2[\cup]1;+\infty[$ Thus, $D_f=S_1\cap S_2=\ ]-\infty;\dfrac{-5}{2}[\cup]1;+\infty[$		
	3	a	$\lim\limits_{x\to 1^-}\dfrac{x^2-4x+3}{x^2-3x+2}=\lim\limits_{x\to 1^-}\dfrac{(x-1)(x-3)}{(x-1)(x-2)}$ $=\lim\limits_{x\to 1^-}\dfrac{x-3}{x-2}$ $=2$	$\lim\limits_{x\to 1^+}\dfrac{1-x}{1-\sqrt{x}}=\lim\limits_{x\to 1^+}\dfrac{1-x}{1-\sqrt{x}}\times\dfrac{1+\sqrt{x}}{1+\sqrt{x}}$ $=\lim\limits_{x\to 1^+}\dfrac{(1-x)(1+\sqrt{x})}{1-\lvert x\rvert}$ $but, x>1$ $=\lim\limits_{x\to 1^+}\dfrac{(1-x)(1+\sqrt{x})}{1-x}$ $=\lim\limits_{x\to 1^+}(1+\sqrt{x})$ $=2$	
		b	f is continuous at $x=1$ if & only if $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{x\to 1^+}f(x)=f(1)$ $f(1)=k^2+k$ And, $\lim\limits_{x\to 1^-}f(x)=2$ (proved)	So, $k^2+k=2$ Or, $k^2+k-2=0$ But, $a+b+c=0$ Thus, $k=\{1,-2\}$	
III	1		If $x=4$ is a root of (E), then replace $x=4$ in (E) to get $(m-1)(4)^2-2(m-2)(4)+m-7=0$, Thus, $m=\dfrac{7}{9}$	For $m=\dfrac{7}{9}$: $(E):-2x^2+22x-54=0$ So, $S=x_1+x_2=-\dfrac{b}{a}=11$, but, $x_1=4$, thus, $x_2=11-4=7$	
	2		(E) admits two distinct roots *if and only if*, $\Delta'=b'^2-ac>0$ $\Delta'=(m-2)^2-(m-1)(m-7)$ $=4m-3$ If, $4m-3>0$ then $m>\dfrac{3}{4}$ Thus, (E) admits two distinct roots for $m\in]\dfrac{3}{4};+\infty[$		

	3	$F=\dfrac{(x_1-1)(x_2-1)}{(x_1^{\ 2}+x_2^{\ 2})}$ $=\dfrac{x_1x_2-(x_1+x_2)+1}{(x_1+x_2)^2-2x_1x_2}$ Now, using (E), $S=\dfrac{-b}{a}=\dfrac{2(m-2)}{m-1}$ & $P=\dfrac{c}{a}=\dfrac{m-7}{m-1}$	So, $F=\dfrac{\dfrac{m-7+2(m-2)+m-1}{m-1}}{4\left(\dfrac{m-2}{m-1}\right)^2-2\left(\dfrac{m-7}{m-1}\right)}$ Thus, $F=\dfrac{2(1-m)}{m^2+1}$	
	4	M_1 is symmetric of M_2 w.r.t. I So, I is the midpoint of $[M_1M_2]$ Then, $x_I=\dfrac{x_{M_1}+x_{M_2}}{2}$	So, $x_1+x_2=2x_I$ But, $x_1+x_2=\dfrac{2(m-2)}{m-1}$ Thus, $m=\dfrac{7}{5}$	
IV	1	Take L.H.S: $\sin^2 3a\cos^2 a-\cos^2 3a\sin^2 a=(\sin 3a\cos a)^2-(\cos 3a\sin a)^2$ $=(\sin 3a\cos a-\cos 3a\sin a)(\sin 3a\cos a+\cos 3a\sin a)$ $=\sin(3a-a)\sin(3a+a)$ $=\sin 2a\sin 4a$ $=\sin 2a(2\sin 2a\cos 2a)$ $=2\cos 2a\sin^2 2a=R.H.S$		
	2	$\sin^2 3a\cos^2 a-\cos^2 3a\sin^2 a=2\cos 2a\sin^2 2a$ And, $E=\dfrac{\sin^2 3a\cos^2 a-\cos^2 3a\sin^2 a}{\cos^3 2a+\cos^2 2a}$ $=\dfrac{2\cos 2a\sin^2 2a}{\cos^2 2a(\cos 2a+1)}$ $=\dfrac{2\sin 2a\tan 2a}{2\cos^2 a}$ Thus, $E=\dfrac{\sin 2a\tan 2a}{\cos^2 a}$		
	3	$E=\dfrac{\sin 2a\tan 2a}{\cos^2 a}$ (proved) where, $\sin 2a=2\sin a\cos a$ & $\tan 2a=\dfrac{2\tan a}{1-\tan^2 a}$ Then , $E=\dfrac{2\sin a\cos a\tan 2a}{\cos^2 a}$ $E=2\tan a\tan 2a$	$\dfrac{E}{2\tan a}=\tan 2a$ Thus, $\dfrac{E}{2\tan a}=\dfrac{2\tan a}{1-\tan^2 a}$	

<table>
<tr><td rowspan="6">V</td><td>1</td><td>For $n=0, u_1=\frac{1}{2}u_0-1=-\frac{1}{2}$
For $n=1, u_2=\frac{1}{2}u_1+2-1=\frac{3}{4}$
For $n=2, u_3=\frac{1}{2}u_2+4-1=\frac{27}{8}$</td><td>Since, $u_2-u_1\neq u_3-u_2$
And, $\frac{u_2}{u_1}\neq\frac{u_3}{u_2}$
Then, (u_n) is neither arithmetic nor geometric.</td><td></td></tr>
<tr><td>2</td><td>For $n=0, v_0=u_0+10=11$
For $n=1, v_1=u_1-4+10=\frac{11}{2}$</td><td>For $n=2, v_2=u_2-8+10=\frac{11}{4}$
For $n=3, v_3=u_3-12+10=\frac{11}{8}$</td><td></td></tr>
<tr><td>3</td><td colspan="2">Since, $\frac{v_1}{v_0}=\frac{v_2}{v_1}=\frac{v_3}{v_2}=\frac{1}{2}$
Thus, (v_n) is a geometric sequence of common ratio $r=\frac{1}{2}$ & $v_0=11$</td><td></td></tr>
<tr><td>4</td><td>(v_n) is a G.S, where $r=\frac{1}{2}$ & $v_0=11$
Then, $v_n=v_0\cdot r^n$</td><td>Thus, $v_n=11\cdot\left(\frac{1}{2}\right)^n$</td><td></td></tr>
<tr><td>5</td><td>$v_n=u_n-4n+10$ (given)
Then, $u_n=v_n+4n-10$</td><td>But, $v_n=11\cdot\left(\frac{1}{2}\right)^n$ (proved)
Thus,
$u_n=11\cdot\left(\frac{1}{2}\right)^n+4n-10$</td><td></td></tr>
<tr><td>6</td><td colspan="2">$T_n=v_0+v_1+v_2+...+v_n$
So, $T_n=\sum_{n=0}^{n}V_n=V_0\left(\frac{1-r^{n+1}}{1-r}\right)=11\left(\frac{1-0.5^{n+1}}{\frac{1}{2}}\right)=22\left(1-0.5^{n+1}\right)$</td><td></td></tr>
<tr><td rowspan="4">VI</td><td>1</td><td colspan="2">$\lim_{x\to 0}\frac{f(x)-f(0)}{x}=f'(0)=1$ $(Graphically)$</td><td></td></tr>
<tr><td>2</td><td colspan="2">$(T): y-f(0)=f'(0)(x-0)$
Thus, $(T): y+2=x$</td><td></td></tr>
<tr><td>3</td><td colspan="2">Table of variations:
<table><tr><td>Values of x</td><td>$-\infty$</td><td></td><td>-1</td><td></td><td>1</td><td></td><td>$+\infty$</td></tr><tr><td>Sign of $f'(x)$</td><td></td><td>+</td><td>0</td><td>+</td><td>0</td><td>-</td><td></td></tr><tr><td>Variation of f</td><td>$-\infty$</td><td>↗</td><td>-4</td><td>↗</td><td>0</td><td>↘</td><td>$-\infty$</td></tr></table></td><td></td></tr>
<tr><td>4</td><td colspan="2">From the above table we notice that: $f(x)<0$ for all $x\in\mathbb{R}^*$</td><td></td></tr>
</table>

VII

1. $\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} x^4\left(1 - \frac{6}{x^2} + \frac{8}{x^3} + \frac{20}{x^4}\right) \to +\infty$

2. $(x-1)^2(x+2) = (x^2 - 2x + 1)(x+2)$

$= x^3 - 3x + 2$

3. $g(x) = x^4 - 6x^2 + 8x + 20$

Thus, $g'(x) = 4x^3 - 12x + 8 = 4(x^3 - 3x + 2) = (x-1)^2(x+2)$

Sign of $g'(x)$ is the sign of $(x-1)^2(x+2)$

But, $(x-1)^2 \geq 0$ for all $x \in \mathbb{R}$ and $(x+2) \geq 0$ for all $x \geq -2$

Thus,

Values of x	$-\infty$		-2		1		$+\infty$
Sign of $g'(x)$		-	0	+	0	+	

4. Table of variations:

Values of x	$-\infty$		-2		1		$+\infty$
Sign of $g'(x)$		$-$	0	+	0	+	
Variation of g	$+\infty$	↘	-4	↗	23	↗	$+\infty$

5. Since,

- $g(x)$ is continuous on $\mathbb{R}$ and $]-3;-2[\subset \mathbb{R}$, so it is continuous on $]-3;-2[$
- $g(x)$ is monotonic on $]-\infty;-2[$ & $]-3;-2[\subset]-\infty;-2[$ so, $g(x)$ is monotonic on $]-3;-2[$

And $g(-3)\cdot f(-2) < 0$

Thus, $g(x) = 0$ admits a unique root $\alpha \in]-3;-2[$

Since,

- $g(x)$ is continuous on $\mathbb{R}$ and $]-2;-1[\subset \mathbb{R}$, so it is continuous on $]-2;-1[$
- $g(x)$ is monotonic on $]-2;1[$ & $]-2;1[\subset]-2;1[$ so, $g(x)$ is monotonic on $]-2;-1[$

And $g(-2)\cdot f(-1) < 0$

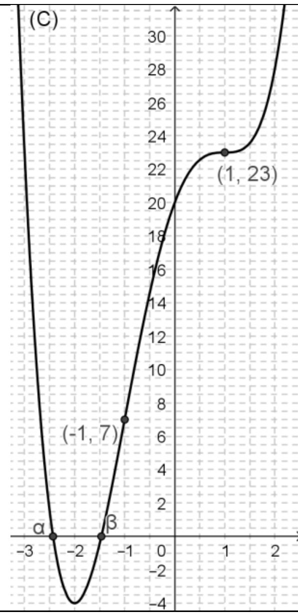
Thus, $g(x) = 0$ admits a unique root $\beta \in]-2;-1[$

6. $g'(x) = 4x^3 - 12x + 8$ (proved)

$g''(x) = -12x^2 - 12 = 12(x^2 - 1)$

Values of x	$-\infty$		-1		$+1$		$+\infty$
Sign of $g''(x)$		+	0	-	0	+	
Concavity of g		∪		∩		∪	

Thus, (C) admits two inflection points $(-1;7)$ & $(1;23)$

	7.		
	8.	Since, $h(x)=g(\|x\|)=\|x\|^4-6\|x\|^2+8\|x\|+20$ So, $h(x)=g(\|x\|)=g(x)$ for all $x\geq 0$ which means, $(C')\equiv(C)$ for all $x\geq 0$ And, $h(-x)=g(\|-x\|)=g(\|x\|)=g(x)$, so $g(\|-x\|)$is even Then, $(C')$is symmetric of (C)w.r.t $y-axis$for all $x<0$