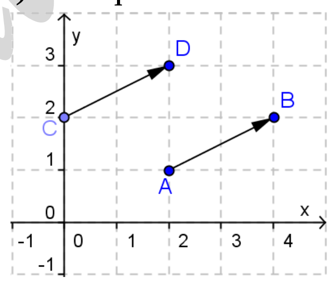
<u>**Def:**</u> In the orthonormal system of axes $(O,\vec{i},\vec{j})$ the ***Scalar product*** of any two vectors $\vec{u}(x;y)$ & $\vec{v}(x';y')$ is a scalar quantity (a real number) given by:

$$\vec{u}.\vec{v} = xx' + yy' \text{ .... } Analytic\ Form.$$

<u>WATCH OUT!</u> The Scalar product of two vectors is in ***independent*** of the choice of the reference frame.

<u>Recall that:</u> Two vectors are equal if they admit the same coordinates.

Ex1: Find the values of m & n, so that the vectors: $\vec{r}(2m-3;5)$ & $\vec{q}(1;3n+2)$ are equal.



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Ex2: Consider in the reference frame $(O,\vec{i},\vec{j})$ the vectors $\vec{u}$ & $\vec{v}$

1- If $\vec{u}=\vec{v}$ then find:
   a. The dot product: $\vec{u}.\vec{v}$ ....
   b. $\|\vec{u}\|$ ....

2- Compare $\vec{u}.\vec{u}$ and $\|\vec{u}\|^2$ ....

Thus, $$\vec{u}.\vec{u} = \vec{u}^2 = \|\vec{u}\|^2 = x^2 + y^2$$

<u>Properties of scalar product:</u>

Ex3: Consider in the reference frame $(O,\vec{i},\vec{j})$ the vectors $\vec{u}(2;-1)$, $\vec{v}(3;4)$ & $\vec{w}(-2;3)$.

Computation	*Comparison*	*Conclusion*
$\vec{u}.\vec{v}$		Scalar product is ***commutative***
$\vec{v}.\vec{u}$		
$\vec{u}.(\vec{v}+\vec{w})$		Scalar product is ***distributive***
$\vec{u}.\vec{v}+\vec{u}.\vec{w}$		

Remarkable scalar products:

In the opposite table, find the other Scalar products.

Naming	*Product form*	*Expanded form*
Square of sum	$\left(\vec{a}+\vec{b}\right)^2$	$\vec{a}^{\,2}+\vec{a}\cdot\vec{b}+\vec{b}^{\,2}$
Square of difference		
Difference between two squares		

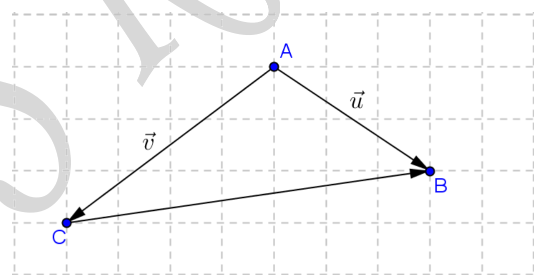
Ex4: Use the above identities to find $\vec{u}.\vec{v}$ in two different ways:

……………………………………… ………………………………………………
……………………………………… ………………………………………………
……………………………………… ………………………………………………

Thus, $\vec{u}.\vec{v}=\frac{1}{2}\left[\left\|\vec{u}+\vec{v}\right\|^2-\left\|\vec{u}\right\|^2-\left\|\vec{v}\right\|^2\right]$ OR $\vec{u}.\vec{v}=\frac{1}{2}\left[\left\|\vec{u}\right\|^2+\left\|\vec{v}\right\|^2-\left\|\vec{u}-\vec{v}\right\|^2\right]$

Use the above results and the following figure to prove that in any triangle we can write that:

$$\overrightarrow{AB}.\overrightarrow{AC}=\frac{1}{2}\left[AB^2+AC^2-BC^2\right]$$



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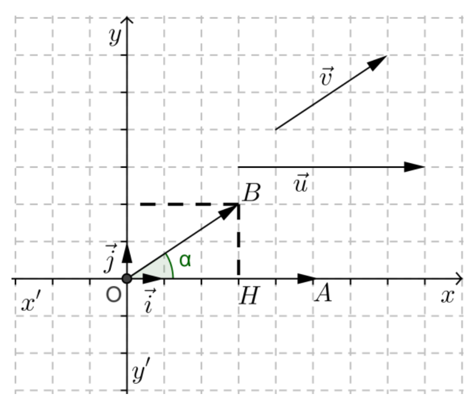
Another form of scalar product:



In the adjacent reference frame $(O,\vec{i},\vec{j})$ we have:

$\overrightarrow{OA}=\vec{u}$ & $\overrightarrow{OB}=\vec{v}$ so that $\overrightarrow{OA}(x;0)$ & $\overrightarrow{OB}(x';y')$

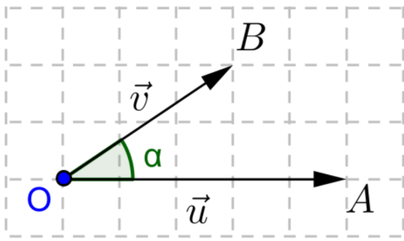
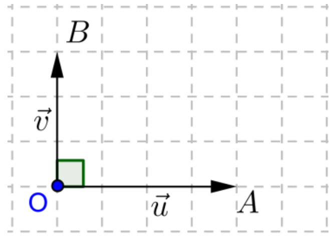
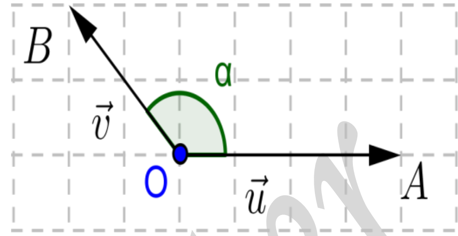
1- Find analytically: $\overrightarrow{OA}.\overrightarrow{OB}=$..................................
2- Let H be orthogonal projection of B on $x-axis$
   - ***a.*** Determine coordinates of $H(....;....)$
   - ***b.*** Find analytically: $\overrightarrow{OA}.\overrightarrow{OH}=$.................................
   - ***c.*** What do you conclude?………………………………..

The ***scalar product*** of any two vectors ***does not change*** if we ***replace*** one of the vectors by its ***orthogonal projection*** on the other (or on an axis of same direction as the other.)

- ***d.*** Find $\overline{OH}$ in terms of $\overline{OB}$ ……………………………………………………
- ***e.*** If $x=\left\|\overrightarrow{OA}\right\|$, then determine, $\overrightarrow{OA}.\overrightarrow{OB}$ again in terms of $\cos\alpha$.

……………………………………………………………………………………

Thus, $\vec{u}\cdot\vec{v}=\left\|\vec{u}\right\|\times\left\|\vec{v}\right\|\cos\alpha$

Sign of a scalar product		
If α is acute $(0 \le \alpha < \frac{\pi}{2})$	If $\alpha = \frac{\pi}{2}$	If α is obtuse $(\frac{\pi}{2} < \alpha \le \pi)$
		
Then, $\cos\alpha > 0$ and $\vec{u}.\vec{v} > 0$	Then, $\cos\alpha = 0$ and $\vec{u}.\vec{v} = 0$	Then, $\cos\alpha < 0$ and $\vec{u}.\vec{v} < 0$

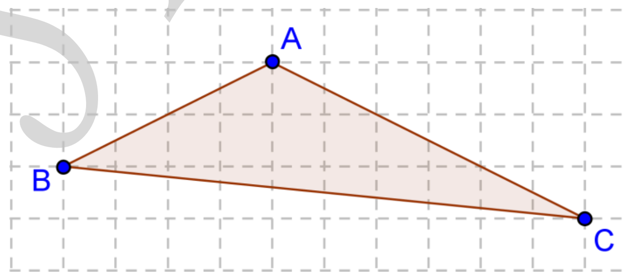
Thus, the Scalar product of two vectors is zero <u>if and only if</u>:

> One of the vectors is zero.
> OR
> The vectors are orthogonal.

Metric Relations in a triangle:

1 - *General form of Pythagoras Theorem:*

Consider ABC to be any triangle.



a) Find $\overrightarrow{AB}.\overrightarrow{AC}$ in terms of measure of sides only.

....

b) Determine $\overrightarrow{AB}.\overrightarrow{AC}$ in terms of $\cos(B\hat{A}C)$.

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c) Deduce the value of BC^2.

....

....

d) Find the value of AC^2 & AB^2.

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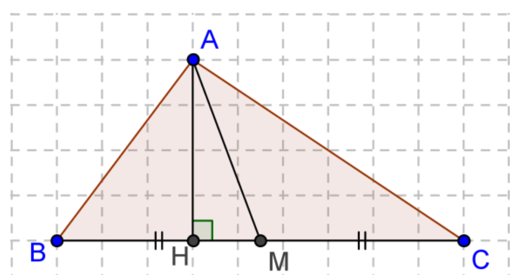
$$BC^2 = AB^2 + AC^2 - 2(AB)(AC)\cos(A\hat{B}C)$$

2 - *Geometric mean:*



$AB^2 - AC^2 = (\overrightarrow{AB} - \overrightarrow{AC}).(\overrightarrow{AB} + \overrightarrow{AC})$

But, $(\overrightarrow{AB} - \overrightarrow{AC}) = \overrightarrow{CB}$ (Chasles' rule)

And, $(\overrightarrow{AB} + \overrightarrow{AC}) = 2\overrightarrow{AM}$ (M is midpoint of $[BC]$)

So, $AB^2 - AC^2 = 2\overrightarrow{CB}.\overrightarrow{AM} = 2\overrightarrow{BC}.\overrightarrow{MA}$

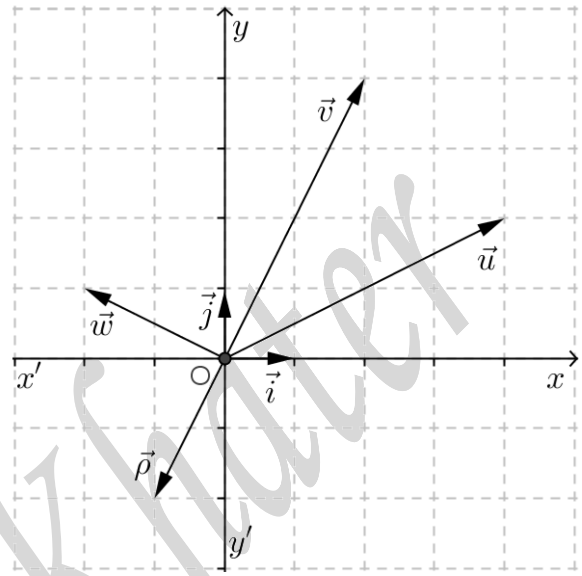
Thus, $$AB^2 - AC^2 = 2CB \times MH$$

Applications

I- Consider the orthonormal system of axes $(O,\vec{i},\vec{j})$:

a. Determine graphically the coordinates of vectors:

$\vec{i}(\ ;\ )$	$\vec{j}(\ ;\ )$	$\vec{w}(\ ;\ )$
$\vec{u}(\ ;\ )$	$\vec{v}(\ ;\ )$	$\vec{\rho}(\ ;\ )$



b. Compute the following scalar (dot) products:

1) $\vec{i}.\vec{i} =$...............................

2) $\vec{i}.\vec{j} =$...............................

3) $\vec{u}.\vec{v} =$...............................

4) $\vec{v}.\vec{w} =$...............................